young researchers to carefully examine the claims of published papers and veteran researchers to review submitted papers more rigorously.

Acknowledgment

The author thanks C. V. Smith Jr. for calling his attention to Ref. 12.

References

[1]Ko, C. L., "Flexural Behavior of a Rotating Sandwich Tapered Beam," *AIAA Journal*, Vol. 27, No. 3, 1989, pp. 359–369.

[2]Ko, C. L., "Dynamic Analysis for Free Vibrations of Rotating Sandwich Tapered Beams," *AIAA Journal*, Vol. 27, No. 10, 1989, pp. 1425–1433.

[3]Bramwell, A. R. S., *Helicopter Dynamics*, Arnold, London, 1976, Chap. 9.

[4]Johnson, W., *Helicopter Theory*, Princeton Univ. Press, Princeton, NJ, 1980, Article 9-3.3.

[5]Stafford, R. O., and Giurgiutiu, V., "Semi-Analytic Methods for Rotating Timoshenko Beams," *International Journal of Mechanical Sciences*, Vol. 17, 1975, pp. 719–727.

[6]Wright, A. D., Smith, C. E., Thresher, R. W., and Wang, J. L. C., "Vibration Modes of Centrifugally-Stiffened Beams," *Journal of Applied Mechanics*, Vol. 49, No. 1, 1982, pp. 197–202.

[7]Mostaghel, N., and Tadjbachsh, I., "Buckling of Rotating Rods and Plates," *International Journal of the Mechanical Sciences*, Vol. 15, June 1973, pp. 429–434.

[8]Peters, D. A., and Hodges, D. H., "In-Plane Vibration and Buckling of a Rotating Beam Clamped Off the Axis of Rotation," *Journal of Applied Mechanics*, Vol. 47, No. 2, 1980, pp. 398–402.

[9]Hodges, D. H., and Dowell, E. H., "Nonlinear Equations of Motion for the Elastic Bending and Torsion of Twisted Nonuniform Rotor Blades," NASA TN D-7818, Dec. 1974.

[10]Kaza, K. R. V., and Kvaternik, R. G., "Nonlinear Flap-Lag-Axial Equations of a Rotating Beam," *AIAA Journal*, Vol. 15, No. 6, 1977, pp. 871–874.

[11]Shames, I. H., and Dym, C. L., *Energy and Finite Element Methods in Structural Mechanics*, Hemisphere, New York, 1985, pp. 197–204.

[12]Smith, C. V., "Comment on 'Large Deflection of Rectangular Sandwich Plates,' " *AIAA Journal*, Vol. 6, No. 3, 1968, p. 574.

[13]Brunelle, E. J., "Stress Redistribution and Instability of Rotating Beams and Disks," *AIAA Journal*, Vol. 9, No. 4, 1971, pp. 758, 759.

[14]Hodges, D. H., "On the Extensional Vibrations of Rotating Bars," *International Journal of Non-Linear Mechanics*, Vol. 12, No. 5, 1977, pp. 293–296.

Reply by the Author to Dewey H. Hodges

C. L. Ko*

Oakland University, Rochester, Michigan 48309-4401

I WOULD like to thank Hodges for his comments on my work, which was presented in Refs. 1 and 2. Hodges considered four items relating to these analyses to be significant errors and indicated that these papers should not have been published. Therefore, I would like to discuss these four issues from a different point of view.

Centrifugal Stiffening Effect

Hodges indicated that governing equations derived in Refs. 1 and 2 were in serious error because the centrifugal stiffening term was absent from those equations. His argument was primarily based on the assumption that the governing equation [Eq. (1) of Hodges' Comment] for a rotating homogeneous beam found in many textbooks, such as Refs. 3 and 4, was absolutely correct. In fact, this particular equation was derived by using the assumption of zero axial deformation and by using a free body of a beam element which was deformed into an arbitrarily sheared element without rotation and curvature as shown in Fig. 9.2 of Ref. 3. Since the bending-curvature relationship used for this derivation was based on Euler–Bernoulli's beam theory, using an arbitrarily sheared element as the free body was actually incorrect because it violated the basic assumption of the theory. As was stated and discussed by Fletcher,[5] the theory assumes that "sections initially planar and perpendicular to the neutral axis remain planar and perpendicular to the deformed neutral axis." A beam element which is consistent with this assumption can only be deformed into a circular differential section with curvature and with rotation but without shear strains. If one uses such an element as the free body to derive equilibrium conditions, governing equations for both the axial and the transverse vibrations will become nonlinear. In addition, terms due to the gravitational force and the static displacement will appear in these equations. Furthermore, the governing equations for the steady-state behavior also will be nonlinear. These complex results also will happen to governing equations for the vibration of nonrotating beams if the same free body is used for the derivation. If both the steady-state and the dynamic nonlinear equations are linearized, the linearization process will imply that both the angle of rotation and the curvature of the element should be neglected. If these two entities are specified to be zero, this deformed beam element will reduce to the same nondeformed free body as those widely utilized in linear mechanics, especially those used for deriving governing equations of nonrotating beams. The linear governing equations for the vibration of rotating beams derived in Refs. 1 and 2 actually are consistent with those derived by applying Newton's mechanics to such a nondeformed free body.

It can be shown that results reported in Ref. 2 were consistent with those obtained by solving governing equations derived by applying Newton's mechanics to a nondeformed free body. The natural frequencies of the beam decrease with increasing rotating speeds for both the axial and the transverse vibrations. If one uses a free body which is consistent with assumptions of Euler–Bernoulli's beam theory and linearizes all governing equations by invoking conflicting assumptions as suggested by Hodges, one can show that the natural frequencies for both the axial and the transverse vibrations are coupled. In addition, frequencies of these coupled modes dominated by the axial vibration can be determined to decrease with increasing rotating speeds. For those modes dominated by the transverse vibration, natural frequencies can be shown to increase with increasing rotating speeds; however, this phenomenon is primarily due to retaining those linear terms which were assumed to be negligibly small in the linearization process. Reference 2 was intended to report results based on a rational analysis consistent with the classical beam theory. If experimental results indicate a completely different behavior for a rotating beam, these results can mean that the classical beam theory is not accurate enough for analyzing the vibration problem of a rotating beam. Regarding the invocation of conflicting assumptions in linearizing the nonlinear equations or using a deformed shape not consistent with the basic assumptions of the classical beam theory as the free body in order to predict experimental results, these approaches can be more irrational mathematically than using the classical linear beam theory. The proper method for this problem might be to solve the complete nonlinear differential equations, such as those derived by using a deformed element with curvature and rotation. However, it is also well known that the nonlinear response of a vibrating structure can be completely different in nature from its linear oscillatory behavior. The major difference lies in the dependency of the natural frequencies on the amplitude of the structure undergoing the nonlinear vibration.

Hodges indicated that there were three proper ways to solve the vibration of rotating beams. In fact, I have found all three methods involve invoking irrational assumptions in their analyses. Therefore, I would like to describe my view of each approach in the same order as they appeared in Hodges' Comment.

Received July 27, 1993; revision received March 16, 1994; accepted for publication Dec. 14, 1994.

*Associate Professor, Department of Mechanical Engineering. Member AIAA.

1) The approach in Ref. 6 used the linear moment-curvature relationship but included geometrically nonlinear terms in the strain-displacement relationships. The linear moment-curvature relationship is actually based on the linear strain-displacement relationships and the small-deflection theory. Using the nonlinear strain-displacement relationships actually implies the beam is subjected to large deformation. Therefore, the linear expression of curvature and the linear moment-curvature relationship become invalid. In addition, it can be shown that the order of the differential equations for describing the coupled axial and transverse vibration will increase if nonlinear strain-displacement relationships are used. Consequently, solutions of these differential equations cannot be obtained due to the problem of having an insufficient number of boundary conditions. Furthermore, neglecting higher order terms and nonlinear terms as was done in Ref. 6 can possibly throw out the first-order effects but retain the second-order effects. Even if this were not the case, the assumption regarding all nonlinear terms being negligible would be equivalent to specifying terms involving the geometric nonlinearity to be actually negligibly small. Unfortunately, Hodges and Dowell[6] did not provide any numerical results to show this method of approach was justified.

2) Kaza and Kvaternik[7] used the geometrically nonlinear axial strain-displacement relationship which was based on the large-deformation theory to describe the kinetic energy of the rotating beam; however, they used the linear bending-curvature relationship derived by using the small-deflection theory for a beam to describe the strain energy. If the geometric nonlinearity is considered, the application of the small-deflection theory to the beam eventually becomes invalid. Furthermore, the linearization process included the second-order effects but excluded the first-order effects. In addition, the linear bending-curvature relationship becomes invalid because it was actually derived by using the linear axial strain-displacement relationship and the small-deflection theory instead of using the nonlinear relationship and the large-deformation theory. Kaza and Kvaternik[7] also used Hamilton's principle to derive the governing equations and then linearized these equations. The "foreshortening" term which was defined as the integration of the geometrically nonlinear terms with respect to the axial coordinate of the beam was added to the expression of the kinetic energy for the beam. Terms in the kinetic-energy expression in Hamilton's principle actually correspond to the acceleration terms in Newton's mechanics. Adding geometrically nonlinear terms to the kinetic energy is equivalent to including additional centripetal nonlinear acceleration due to the geometric nonlinearity using Newton's mechanics. Since the centripetal acceleration is linearly proportional to the axial displacement of the particle considered, this nonlinear acceleration actually cannot be obtained by using Newton's mechanics. Therefore, Kaza and Kvaternik's nonlinear expression[7] of the kinetic energy can be questionable. Furthermore, the additional elongation due to the geometric nonlinearity should be more significant in affecting the strain energy than affecting the kinetic energy. In addition, in the linearization process, conflicting assumptions involving orders of magnitude of the nonlinear and the linear terms also were invoked.

3) In Ref. 3, the virtual work done by the centrifugal tension in the blade was included in the strain-energy expression. Unfortunately, the expression of the radial elongation was also based on the same arbitrarily sheared element as that shown in Fig. 9.2 of Ref. 3. This expression of the radial displacement for determining the centrifugal tension actually is not compatible with the expression of the strain energy due to bending. Euler–Bernoulli's moment-curvature relationship, which was used in the expression, became invalid if the radial elongation was expressed as that used for determining the centrifugal tension. Therefore, the energy method used in Ref. 3 has the same problem as the application of Newton's mechanics to the free body of an arbitrarily sheared element as shown in Fig. 9.2 of the same reference.

In summary, all three methods assumed that the textbook equation [Eq. (1) of Hodges' Comment] was absolutely correct and tried to use wrong expressions for strains and displacements to match this equation. This equation was derived from application of the small-deflection linear beam theory to an unreasonable free body; however, methods discussed in items 1 and 2 even adopted a different type of physical principle, namely, the large deformation theory to match this erroneous equation. The linear analysis shown in Refs. 1 and 2 might not be perfect, but at least this analysis is consistent with the classical beam theory. Since the numerical results obtained by using this linear analysis might not match the experimental measurements, a better method which can be consistent with the basic physics of rotating beams should, therefore, be researched further.

Total Section Rotation Variable

Hodges indicated that kinematical boundary conditions at the fixed end of Timoshenko beams used in Refs. 1 and 2 were in serious error. This argument was based on their being different from those derived by Shames and Dym.[8] The analysis shown by Shames and Dym[8] considered the deflection function w_0 to be independent of the rotation parameter $\theta \equiv \psi + (\partial w_0/\partial x)$, whereas the approach used in Refs. 1 and 2 treated the deflection function w_0 to be independent of the shear strain $(-\psi)$ instead of the rotation parameter $\theta \equiv \psi + (\partial w_0/\partial x)$ because w_0 and ψ were defined as the transverse displacement of the neutral axis and the shear rotation about this axis, respectively. Even though governing equations obtained by using these two different approaches appear to be defferent in form, they can be manipulated to yield identical equations. Using the latter approach in the variational analysis, one can obtain $\delta\psi = 0$ and $\delta(\partial w_0/\partial x) = 0$ as two of the four kinematic boundary conditions at the fixed end of the beam. Although the condition of identifying the shear strain at the fixed end was written as $\psi = 0$ instead of $\delta\psi = 0$, the shear strain at the fixed end was actually specified as those shown in Eq. (69) of Ref. 1. In fact, the expression of $\psi = 0$ is only meant to specify the integration constant in Eq. (69) to be zero. Regarding the same expression used in Ref. 2, it was never used in the analysis because three governing equations were combined into two by eliminating ψ from these equations. Therefore, the shear parameter ψ was specified to be zero at the fixed end for simplicity. Its actual value at the fixed end cannot easily be determined for a vibration problem. I have to admit that the expression $\psi = 0$ did not reflect the actual condition at the clamped end and could be very confusing to readers.

Regarding whether or not the deflection slope, $\partial w_0/\partial x$, should be zero at the fixed end, Timoshenko[9] actually indicated that two possible solutions could exist for a cantilever beam. One solution was based on the assumption that the built-in cross section could warp freely and the other was based on the assumption that the built-in cross section was completely prevented from warping. Timoshenko's solution[9] for the case of allowing warping to occur at the fixed end was not based on the requirement of satisfying the zero-slope condition. Instead, he considered the deflection at the free end of a cantilever beam subjected to a tip load to be the same as the deflection at the center of a simply supported beam with twice the length and twice the load being applied at the center. Since the bending moment of a cantilever beam is in the opposite sense to that of a simply supported beam and their shear forces are in the same direction, the shear contribution to bending moments and deflections of these beams should be additive for the simply supported beam and substractive for the cantilever beam. If Timoshenko[9] had taken this phenomenon into consideration, his result for the case of allowing warping to occur would have been the same as the solution obtained by specifying the deflection slope to be zero at the fixed end. References 1 and 2 took the similar approach of allowing the built-in cross section to warp by forcing the slope of the deflection curve to be zero at the fixed end. If the deflection slope is specified to be zero at the fixed end, the axial displacement at this end expressed in Eq. (16) of Ref. 1 will become $u = -y\psi$. This situation is identical to allowing warping to occur at the fixed end. I admit that it is physically more reasonable to prevent the built-in end to warp. However, preventing warping completely at the fixed end also can imply that the slope of the neutral axis has a sudden discontinuity from being zero inside the built-in cross section to a nonzero value at the fixed end of the beam. Therefore, it is also not entirely irrational to allow the built-in cross section to warp because the warping can be considered the natural response of the beam to the end moment for holding the neutral axis in the level position inside the built-in cross section.

Axial Instability

Hodges' argument regarding the axial instability can also be applied to the transverse instability as well. As was discussed in Ref. 1, the maximum stress in a rotating steel or aluminum beam normally exceeds its yield strength at a rotating speed much lower than any of the unstable rotating speeds for these two instabilities. Similar to Brunelle's analysis,[10] the discussion of inertioelastic instabilities in Ref. 1 was intended only to show a mathematical prediction based on the linear theory rather than to indicate instabilities that actually exist in rotating metallic beams.

Dissipative Forces

Regarding whether or not the variational principle can be applied to cases with nonconservative forces, Goldstein[11] provided some history of this subject. For a system with only dissipative forces and no potential function, the Lagrangian was expressed by Goldstein[11] as

$$L = T - U \tag{1}$$

where U was called a "general potential" or a "velocity-dependent potential" by Goldstein.[11] According to Goldstein,[11] the German mathematician Schering seems to have been the first to seriously attempt to include velocity-dependent forces in the framework of mechanics in 1873. Goldstein[11] suggested the name "generalized potential" to include ordinary potential energy within this designation. The time integration of the virtual work expressed in Ref. 1 actually is a result of considering such a generalized potential in the variational principle.

The dissipative forces considered in Ref. 1 were viscous damping forces in the structure. Terms involving Ω in Eq. (36) did not appear in the governing equations, Eqs. (40–42). Since these governing equations were derived to include both the steady-state analysis and the vibration analysis, general expressions of the velocity components of particles in the structure were used in Eq. (36) to indicate the damping force is being related to the particle velocity. To be more precise, for the case of describing vibration only, these terms should have been left out in Eq. (36). However, excluding them does not change the governing equations, Eqs. (40–42), shown in Ref. 1.

References

[1]Ko, C. L., "Flexural Behavior of a Rotating Sandwich Tapered Beam," *AIAA Journal*, Vol. 27, No. 3, 1989, pp. 359–369.

[2]Ko, C. L., "Dynamic Analysis for Free Vibrations of Rotating Sandwich Tapered Beams," *AIAA Journal*, Vol. 27, No. 10, 1989, pp. 1425–1433.

[3]Bramwell, A. R. S., *Helicopter Dynamics*, Arnold, London, 1976.

[4]Johnson, W., *Helicopter Theory*, Princeton Univ. Press, Princeton, NJ, 1980.

[5]Fletcher, D. Q., *Mechanics of Materials*, CBS College, New York, 1985, p. 211.

[6]Hodges, D. H., and Dowell, E. H., "Nonlinear Equations of Motion for the Elastic Bending and Torsion of Twisted Nonuniform Rotor Blades," NASA TN D-7818, 1974.

[7]Kaza, K. R. V., and Kvaternik, R. G., "Nonlinear Flap-Lag-Axial Equations of a Rotating Beam," *AIAA Journal*, Vol. 15, No. 6, 1977, pp. 871–874.

[8]Shames, I. H., and Dym, C. L., *Energy and Finite Element Methods in Structural Mechanics*, Hemisphere, New York, 1985, pp. 197–204.

[9]Timoshenko, S., *Strength of Materials, Part I*, Van Nostrand, New York, 1940, pp. 186–191.

[10]Brunelle, E. J., "Stress Redistribution and Instability of Rotating Beams and Disks," *AIAA Journal*, Vol. 9, No. 4, 1971, pp. 758, 759.

[11]Goldstein, H., *Classical Mechanics*, Addison-Wesley, Reading, MA, 1981, p. 21.

Comment on "Application of Singular Value Decomposition to Direct Matrix Update Method"

Alex Berman*
Bloomfield, Connecticut 06002

IN a recent Technical Note by To,[1] he investigated the mass matrix correction method of Ref. 2. His conclusions may be briefly summarized as follows: the matrix representing the change in the mass will have a rank equal to the lower of the number of modes used or the number of mass error sites. I find no fault with the analysis as presented and I am, in fact, pleased with the results of this study. I believe, however, that I should comment on some common misconceptions of the applicability of model updating methods which are in this paper[1] and in many others.

In studies of such techniques, as in Ref. 1, it is common to treat the structure as a finite set of lumped masses connected by linear springs. It is then possible to consider complete and incomplete sets of measured modes. This concept was also implied by myself in Ref. 2. When the method was applied to a real problem[3] and enhanced by other relevant techniques, such as, from Ref. 4, this was found to be an inappropriate oversimplification. This and other related considerations are discussed in some detail in Refs. 5 and 6. A very brief summary of this important, but rarely acknowledged, problem follows.

A real structure has essentially an infinite number of degrees of freedom. A complete finite element representation of a linear real structure must be nonlinear. For a real structure, there are an infinite number of "good" linear models having a finite number of degrees of freedom.[6] Each of these models has a limited range of applicability. Only a subset of the modes of the model will correspond to modes of the structure. Thus, the concept that one needs "all" the modes for a procedure to give the exact answer makes no sense at all.

The "improved" mass matrix of Refs. 2 and 3 is not a "correct" matrix. It is one of the many which will yield an analytical model which will behave like the structure within the limited applicable range. Since it also deviates by some minimum amount from a "pretty good" analytically derived model, one may expect that it also has some applicability over a somewhat wider range than the test data. Further studies are required in this area.

To[1] uses as his structure a finite system with a finite number of modes. He demonstrates, that for this system, the method of Ref. 2 will work when using less than all the modes, if the number of modes is at least equal to the number of mass error sites. Although his conclusion is appropriate for the problem he presented, unfortunately, it does not apply to the problem of improving an analytical model of a physical structure.

References

[1]To, W. M., "Application of Singular Value Decomposition to Direct Matrix Update Method," *AIAA Journal*, Vol. 32, No. 10, 1994, pp. 2124–2126.

[2]Berman, A., "Mass Matrix Correction Using an Incomplete Set of Measured Modes," *AIAA Journal*, Vol. 17, No. 10, 1979, pp. 1147–1148.

[3]Berman, A., and Nagy, E. J., "Improvement of a Large Analytical Model Using Test Data," *AIAA Journal*, Vol. 21, No. 8, 1983, pp. 1168–1173.

[4]Baruch, M., and Bar Itzhak, I. Y., "Optimal Weighted Orthogonalization of Measured Modes," *AIAA Journal*, Vol. 16, 1978, pp. 346–351.

[5]Berman, A., "System Identification of Structural Dynamic Models—Theoretical and Practical Bounds," AIAA Paper 84-0929, May 1984.

[6]Berman, A., "Multiple Acceptable Solutions in Structural Model Improvement," *AIAA Journal* (to be published).

Received Oct. 31, 1994; revision received Dec. 14, 1994; accepted for publication Dec. 14, 1994.

*Consulting Engineer, 75 Kenwood Circle. Member AIAA.